Characterization of bipartite states using a single homodyne detector

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Abstract. We suggest a scheme to reconstruct the covariance matrix of a two-mode state using a single homodyne detector plus a polarizing beam splitter and a polarization rotator. It can be used to fully characterize bipartite Gaussian states and to extract relevant informations on generic states.

1. Introduction

Bipartite (entangled) states of two modes of the radiation field are the basic tool of quantum information processing with continuous variables [1, 2, 3].

Bipartite states can be produced by different schemes, mostly based on parametric processes in active nonlinear optical media. Generation schemes are either Hamiltonian two-mode processes, like parametric downconversion [4] or mixing of squeezed states [5], or conditional schemes based on the generation of multipartite states followed by conditional measurements [6].

Besides mean values of the field operators, the most relevant quantity needed to characterize a bipartite state is its covariance matrix. For Gaussian states, a class that encompasses most of the states actually realized in quantum optical labs, the first two moments fully characterize the quantum state [7, 8]. Once the covariance matrix is known then the entanglement of the state can be evaluated and, in turn, the performances of the state itself in serving as a support for quantum information protocols like teleportation or dense coding.

Entanglement is generally corrupted by the interaction with the environment. Therefore, entangled states that are available for experiments are usually mixed states, and it becomes crucial to establish whether or not entanglement has survived the environmental noise. As a consequence, besides being of fundamental interest, a simple characterization technique for bipartite states is needed for experimentally check the accessible entanglement in a noisy channel [9, 10, 11, 12, 13], as well as the corresponding state purity and nonclassicality [14, 15].

In this paper we suggest a scheme to measure the first two moments of a bipartite state using repeated measurements of single-mode quadratures made with a single homodyne detector. This is an improvement compared to the scheme of Ref. [13], where two homodyne detectors have been employed. The scheme involves fourteen quadratures pertaining to five different field modes. It can be used to fully characterize bipartite Gaussian states or to extract relevant informations on a generic state.

In the next Section we introduce the notation and describe how to obtain the mean values and the covariance matrix starting from the statistics of suitably chosen field quadratures. In Section 3 a possible experimental realization is described in details. Section 4 closes the paper with some concluding remarks.

2. Bipartite Gaussian states and reconstruction of the covariance matrix

Our scheme is aimed to reconstruct the first two moments of a bipartite states. This represents a relevant piece of information on any quantum state of two modes and provide the full characterization of the quantum state in the case of Gaussian signals. Gaussian states, *i.e.* states with a Gaussian characteristic function, are at the heart of quantum information processing with continuous variables. The basic reason is that the vacuum state of quantum electrodynamics is itself a Gaussian state. This observation, in combination with the fact that the quantum evolutions achievable with current technology are described by Hamiltonian operators at most bilinear in the quantum fields, accounts for the fact that the states commonly produced in laboratories are Gaussian. In fact, bilinear evolutions preserve the Gaussian character of the vacuum state [16]. Furthermore, recall that the operation of tracing out a mode from a multipartite Gaussian state preserves the Gaussian character too, and the same observation is valid when the evolution of a state in a standard noisy channel is considered.

We denote the two modes under investigation by a and b. In the following we assume that a and b have equal frequencies and different polarizations. The Cartesian operators q_k and p_k, $k = a, b$ can be expressed in terms of the mode operators as follows

$$q_a = \frac{1}{\sqrt{2}}(a^\dagger + a), \qquad p_a = \frac{i}{\sqrt{2}}(a^\dagger - a), \tag{1}$$

and analogously for q_b and p_b. The covariance matrix of a two-mode state is a real symmetric positive matrix defined as follows

$$\boldsymbol{\sigma} = \begin{pmatrix} \Delta q_a^2 & \Delta q_a p_a & \Delta q_a q_b & \Delta q_a p_b \\ \Delta p_a q_a & \Delta p_a^2 & \Delta p_a q_b & \Delta p_a p_b \\ \Delta q_b q_a & \Delta q_b p_a & \Delta q_b^2 & \Delta q_b p_b \\ \Delta p_b q_a & \Delta p_b p_a & \Delta p_b q_b & \Delta p_b^2 \end{pmatrix}, \tag{2}$$

where $\Delta X^2 = \langle X^2 \rangle - \langle X \rangle^2$ and $\Delta XY = \frac{1}{2}\langle [X, Y]_+ \rangle - \langle X \rangle \langle Y \rangle$ denote the variance of the observable X and the and mutual correlations between the observables X and Y

respectively. $[X, Y]_+ = XY + YX$ denotes the anticommutator between the operators X and Y. Throughout the paper $\langle X \rangle$ will denote the ensemble average $\langle X \rangle = \text{Tr}\,[R\,X]$, R being the density matrix describing the two-mode state. The characteristic function of a quantum state R is defined as the expectation values $\chi(\lambda_1, \lambda_2) = \langle D(\lambda_1) \otimes D(\lambda_2) \rangle$ where $\lambda_j \in \mathbb{C}$, $j = 1, 2$ and $D(\lambda) = \exp\{\lambda a^\dagger - \lambda^* a\}$ is the displacement operator. The most general bipartite Gaussian state corresponds to a characteristic function of the form

$$\chi(\boldsymbol{\lambda}) = \exp\left\{-\tfrac{1}{2}\boldsymbol{\lambda}^T \boldsymbol{\sigma} \boldsymbol{\lambda} - i \boldsymbol{\lambda}^T \boldsymbol{X}\right\}, \tag{3}$$

where $\boldsymbol{\lambda} = (\text{Re}\,[\lambda_1], \text{Im}[\lambda_1], \text{Re}\,[\lambda_2], \text{Im}[\lambda_2])^T$ and $(\cdots)^T$ denotes transposition. The vector $\boldsymbol{X} = (\langle q_a \rangle, \langle p_a \rangle, \langle q_b \rangle, \langle q_b \rangle)^T$ contains the mean value of the Cartesian mode operators. The characteristic function fully specify a quantum state, *i.e.* any expectation value may be obtained as a phase space integral. Since for a Gaussian state the first two moments specify the characteristic function, their knowledge fully characterize a bipartite Gaussian state.

2.1. Covariance matrix from quadrature measurement

For the sake of simplicity, we rewrite the covariance matrix as follows:

$$\boldsymbol{\sigma} = \boldsymbol{V} - \boldsymbol{M} \tag{4}$$

where the variance $\boldsymbol{V}$ and the mean $\boldsymbol{M}$ matrices may be written as

$$\boldsymbol{V} = \begin{pmatrix} \langle q_a^2 \rangle & \tfrac{1}{2}\langle [p_a, q_a]_+ \rangle & \langle q_a q_b \rangle & \langle q_a p_b \rangle \\ \tfrac{1}{2}\langle [p_a, q_a]_+ \rangle & \langle p_a^2 \rangle & \langle p_a q_b \rangle & \langle p_a p_b \rangle \\ \langle q_b q_a \rangle & \langle q_b q_a \rangle & \langle q_b^2 \rangle & \tfrac{1}{2}\langle [q_b, p_b]_+ \rangle \\ \langle p_b q_a \rangle & \langle p_b p_a \rangle & \tfrac{1}{2}\langle [p_b, q_b]_+ \rangle & \langle p_b^2 \rangle \end{pmatrix}, \tag{5}$$

and

$$\boldsymbol{M} = \begin{pmatrix} \langle q_a \rangle^2 & \langle p_a \rangle \langle q_a \rangle & \langle q_a \rangle \langle q_b \rangle & \langle q_a \rangle \langle p_b \rangle \\ \langle p_a \rangle \langle q_a \rangle & \langle p_a \rangle^2 & \langle p_a \rangle \langle q_b \rangle & \langle p_a \rangle \langle p_b \rangle \\ \langle q_b \rangle \langle q_a \rangle & \langle q_b \rangle \langle q_a \rangle & \langle q_b \rangle^2 & \langle q_b \rangle \langle p_b \rangle \\ \langle p_b \rangle \langle q_a \rangle & \langle p_b \rangle \langle p_a \rangle & \langle p_b \rangle \langle q_b \rangle & \langle p_b \rangle^2 \end{pmatrix}. \tag{6}$$

Once defined the quadrature operator of the mode k, namely

$$x_{k,\phi} = \frac{k^\dagger\, e^{i\phi} + k\, e^{-i\phi}}{\sqrt{2}}, \tag{7}$$

we use the following conventions:

$$x_k \equiv x_{k,0}, \quad y_k \equiv x_{k,\pi/2}, \tag{8a}$$

$$z_k \equiv x_{k,\pi/4}, \quad t_k \equiv x_{k,-\pi/4}. \tag{8b}$$

The matrix $\boldsymbol{M}$ only contains the first moments and can be reconstructed by measuring the four quadratures x_k and y_k, $k = a, b$. We have

$$\langle q_k \rangle = \langle x_k \rangle, \quad \langle p_k \rangle = \langle y_k \rangle. \tag{9}$$

In order to reconstruct the variance matrix $\boldsymbol{V}$ more quadratures are needed. Let us introduce the modes

$$a, \quad b, \quad c = \frac{a+b}{\sqrt{2}}, \quad d = \frac{a-b}{\sqrt{2}}, \quad e = \frac{ia+b}{\sqrt{2}}, \quad f = \frac{ia-b}{\sqrt{2}}. \tag{10}$$

If a and b correspond to vertical and horizontal polarizations, then c and d are rotated polarization modes at $\pm\pi/4$, whereas e and f correspond to left- and right-handed circular polarizations. After tedious but straightforward calculations, we have:

$$\boldsymbol{V} = \frac{1}{2} \begin{pmatrix} 2\langle x_a^2 \rangle & \langle z_a^2 \rangle - \langle t_a^2 \rangle & \langle x_c^2 \rangle - \langle x_d^2 \rangle & \langle y_e^2 \rangle - \langle y_f^2 \rangle \\ \langle z_a^2 \rangle - \langle t_a^2 \rangle & 2\langle y_a^2 \rangle & \langle x_f^2 \rangle - \langle x_e^2 \rangle & \langle y_c^2 \rangle - \langle y_d^2 \rangle \\ \langle x_c^2 \rangle - \langle x_d^2 \rangle & \langle x_f^2 \rangle - \langle x_e^2 \rangle & 2\langle x_b^2 \rangle & \langle z_b^2 \rangle - \langle t_b^2 \rangle \\ \langle y_e^2 \rangle - \langle y_f^2 \rangle & \langle y_c^2 \rangle - \langle y_d^2 \rangle & \langle z_b^2 \rangle - \langle t_b^2 \rangle & 2\langle y_b^2 \rangle \end{pmatrix}. \tag{11}$$

Furthermore, since

$$\boldsymbol{V}_{14} = \boldsymbol{V}_{41} = \tfrac{1}{2}\left(\langle y_e^2 \rangle - \langle y_f^2 \rangle\right) = \langle y_e^2 \rangle - \tfrac{1}{2}\left(\langle x_a^2 \rangle + \langle y_b^2 \rangle\right), \tag{12a}$$

$$\boldsymbol{V}_{23} = \boldsymbol{V}_{32} = \tfrac{1}{2}\left(\langle x_f^2 \rangle - \langle x_e^2 \rangle\right) = \tfrac{1}{2}\left(\langle x_b^2 \rangle + \langle y_a^2 \rangle\right) - \langle x_e^2 \rangle, \tag{12b}$$

the measurement of the quadratures pertaining to mode f is not essential. Overall, in our scheme, the reconstruction of the covariance matrix requires the measurement of at least fourteen quadratures, *e.g.* the following ones (of course measuring also the f-quadratures, being additional independent measurements, would improve the accuracy of the reconstruction)

$$\begin{array}{cccc} x_a, & y_a, & z_a, & t_a, \\ x_b, & y_b, & z_b, & t_b, \\ x_c, & y_c, & x_d, & y_d, \\ x_e, & y_e; \end{array}.$$

Notice that the number of parameters needed to characterize a bipartite Gaussian state is also equal to fourteen.

3. Experimental implementations

In Section 2 we have proved that it is possible to fully reconstruct the covariance matrix $\boldsymbol{\sigma}$ by measuring fourteen different quadratures of five field modes obtained as linear combination of the initial pair. Here we consider an implementation based on the bright continuous-wave beams generated by a seeded degenerate optical parametric amplifier
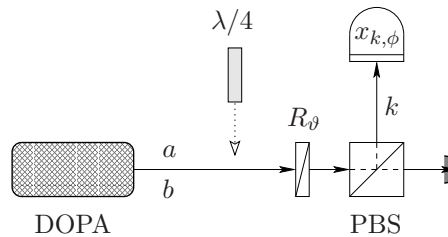
Figure 1. Scheme of a possible apparatus to measure the covariance matrix of the bipartite (entangled) state generated by a DOPA. The two modes, a (vertical polarization) and b (horizontal polarization), pass through a (removable) $\lambda/4$ wave-plate and a rotator of polarization R_ϑ; finally, a PBS reflects the vertically polarized component of its input toward a homodyne detector, which measures the $x_{k,\phi}$ quadrature. See text for details.

(DOPA) below threshold based on a type–II nonlinear crystal [17]. The two collinear beams (a and b) exiting the DOPA are orthogonally polarized and excited in a continuous variable bipartite entangled state. In the following we assume a as vertically polarized and b as horizontally polarized.

Since the mode f is not necessary to reconstruct the covariance matrix, we do not consider its selection, focusing our attention on modes a, b, c, d, and e. The mode under scrutiny is selected by inserting suitable components on the optical path of fields a and b, before the homodyne detector. Modes a, b, c, and d are obtained by means of a rotator of polarization R_ϑ (namely a $\lambda/2$ wave-plate) and a polarizing beam splitter (PBS), which reflects toward the detector the vertically polarized component of the impinging beam. The action of the rotator R_ϑ on the basis $\{|V\rangle, |H\rangle\}$ is given by

$$R_\vartheta|V\rangle = \cos\vartheta\,|V\rangle - \sin\vartheta\,|H\rangle\,, \tag{13a}$$

$$R_\vartheta|H\rangle = \sin\vartheta\,|V\rangle + \cos\vartheta\,|H\rangle\,. \tag{13b}$$

In order to select mode e a $\lambda/4$ wave-plate should be inserted just before the rotator R_ϑ (see Fig. 1). The $\lambda/4$ wave-plate produces a $\pi/2$ shift between horizontal and vertical polarization components, thus turning the polarization from linear into circular.

Table 1 summarizes the settings needed to select the five modes. Overall, the vertically polarized mode k arriving at the detector can be expressed in terms of the initial modes as follows

$$k = \exp\{i\varphi\}\cos\vartheta\,a + \sin\vartheta\,b\,, \tag{14}$$

where $\varphi = \pi/2$ when the $\lambda/4$ wave-plate is inserted, $\varphi = 0$ otherwise.

Once the mode k has been selected, a homodyne detector is used to measure the generic quadrature $x_{k,\phi}$. Homodyne relies on the controlled interference between the quantum beam (signal) to be analyzed and a strong "classical" local oscillator (LO) beam of phase ϕ. Indeed, to access $x_{k,\phi}$ one have to suitably tune the phase ϕ. The optimization of the efficiency is provided by matching the LO mode to the mode k. The mode matching requires precise control of the LO frequency, spatial and polarization

Mode	$\lambda/4$	R_ϑ
a	no	0
b	no	$+\pi/2$
c	no	$+\pi/4$
d	no	$-\pi/4$
e	yes	$+\pi/4$

Table 1. Setting to select the different modes k. The table refers to the elements depicted in Fig. 1. The mode a is assumed to be vertically polarized and the mode b horizontally polarized.

properties. Remarkably, the detected mode is always vertically polarized, thus avoiding any need of tuning the LO polarization.

4. Conclusions

A simple scheme has been suggested to reconstruct the covariance matrix of two-mode states of light using a single homodyne detector plus a polarizing beam splitter and a polarization rotator. Our scheme requires the local measurements of 14 different quadratures pertaining to five field mode. It can be used to fully characterize bipartite Gaussian states and to extract relevant informations on generic states. Finally, we notice that an efficient source of polarization squeezing has been recently realized [18], which might be considered as a preliminary stage for the experimental realization of the present characterization scheme.

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